ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 111–127

q-ANALOGS OF HARMONIC OSCILLATORS AND RELATED RINGS

BY

ELLEN E. KIRKMAN

Department of Mathematics
Box 7311 Wake Forest University
Winston–Salem, NC 27109
email: kirkman@mthcsc.wfu.edu

AND

LANCE W. SMALL

Department of Mathematics 0112
University of California, San Diego
La Jolla, CA 92093-0112
email: lwsmall@ucsd.edu

ABSTRACT

A ring H_q which is a q–analog of the universal enveloping algebra of the Heisenberg Lie algebra $U(h)$ is constructed, and its ring theoretic properties are studied. It is shown that H_q has a factor ring A_q which is a simple domain with properties that are compared to the Weyl algebra. A second q–analog H_q of $U(h)$ is constructed, and H_q is shown to be a primitive ring.

The three dimensional Heisenberg Lie algebra h, with basis $\{x, y, z\}$ and relations $[y, x] = z$, $[x, z] = 0$, and $[y, z] = 0$ is a nilpotent Lie algebra which occurs in quantum mechanics in the solution of the "harmonic oscillator problem." More precisely, the "oscillator representation" of the Weyl algebra $A_1(\mathbb{R}) \simeq U(h)/\langle z-1\rangle$ gives the solution to the harmonic oscillator problem (some physicists do not distinguish the rings $U(h)$ and $A_1(\mathbb{R})$ carefully, perhaps because they can set a central element equal to one by a change of units). Here we consider ring theoretic properties of "q-analogs" of the Weyl algebra $A_1(\mathbb{C})$ and of the

Received July 18, 1991

universal enveloping algebra of the complex Heisenberg algebra $U(h)$. In the first section we consider q-analogs which arise from the work of physicists in constructing a q-analog to the quantum harmonic oscillator ([B], [H], [M]). In the second section we consider another q-analog of $U(h)$, and we shall prove some results of independent interest concerning primitive rings and simple skew polynomial rings. We shall show that our q-analogs of $U(h)$ are similar to $U(h)$ in that they are graded, regular (in the sense of Artin–Schelter [AS]), Noetherian domains of global dimension three, but different than $U(h)$ in that they have primitive factor rings which are not simple, and they are not catenary rings. We will see that a factor ring of our q-analog to $U(h)$, the ring A_q of [H], provides a reasonable analog of the Weyl algebra.

Much work has been done in producing useful q-analogs of the universal enveloping algebra of a complex semisimple Lie algebra (see e.g. [L1], [R], [S]); these q-analogs have a noncommutative, non-cocommutative Hopf algebra structure which makes them "quantum groups" or "quantized universal enveloping algebras." The q-analogs related to the nilpotent Lie algebra h which we investigate here have no apparent Hopf structure, yet they have interesting relationships to the quantum groups which have been previously studied.

Both the Weyl algebra $A_1(\mathbb{C})$ and the universal enveloping algebra of the Heisenberg algebra $U(h)$ are skew polynomial rings; the q-analogs which we consider are also skew polynomial rings, but with a nontrivial automorphism and a skewed derivation. Recall that the skew polynomial ring $R[X;\sigma,\delta]$ is the set of elements of the form $\sum_{i=0}^{n} a_iX^i$ for $a_i \in R$, and that multiplication is defined by $Xa = \sigma(a)X + \delta(a)$, where σ is a ring endomorphism of R and δ is a σ-derivation of R (i.e. δ is an additive endomorphism of R with $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$). The Weyl algebra $A_1(\mathbb{C})$ is $\mathbb{C}[X][Y;\sigma,\delta]$ where σ is the identity map and δ is the usual derivative on $\mathbb{C}[X]$. The enveloping algebra $U(h)$ is $\mathbb{C}[X,Z][Y;\sigma,\delta]$ where σ is the identity map, $\delta(X) = Z$, and $\delta(Z) = 0$. We shall call a σ-skew derivation δ an **inner derivation** if $\delta = \delta_a$ for some $a \in R$, where $\delta_a(x) = ax - \sigma(x)a$.

We shall also make use of the notion of a G-ring. Recall that R is a G**-ring** if R is a prime ring in which the intersection of nonzero prome ideals is nonzero. If R has a normalizing element c such that R localized at the powers of c is a simple ring, then R is a G-ring. It is not hard to see that a semiprimitive G-ring is primitive.

ACKNOWLEDGEMENT: This work was done while the first author was visiting the Universitiy of California, San Diego. She wishes to thank the department for its hospitality. We also wish to thank physicists T. Kirkman (St. Johns University) and J. Rabin (UCSD) for some useful conversations.

The research of both authors was supported in part by grants from the National Science Foundation.

1. The quantum harmonic oscillator problem

The usual harmonic oscillator problem in quantum mechanics is to find operators a^+ (the creation operator) and a (the annihilation operator) (where a^+ is the transpose of a) acting on a Hilbert space with orthonormal basis $\{v_n : n = 0, 1, \ldots\}$ so that $[a, a^+] = 1$ and $Hv_n = (n + (1/2))\hbar w v_n$, where H is the Hamiltonian, $H = \hbar w(a^+a + (1/2))$. The matrices representing a and a^+ act on $\{v_n\}$ so that $a^+v_n = \sqrt{(n+1)}v_{n+1}$, $av_n = \sqrt{n}v_{n-1}$, $av_0 = 0$; these matrices give the "oscillator representation" of the Weyl algebra $A_1(\mathbb{R})$ (i.e. $V = \{v_n\}$ is a faithful irreducible module for the simple ring $A_1(\mathbb{R})$, where $X \leftrightarrow a^+$ and $Y \leftrightarrow a$ act on V). (Physicists work over $\mathbb{R}$; for our purposes $\mathbb{C}$ works just as well).

Goodearl [G] and Morikawa [Mo] have proposed as a q-analog to the Weyl algebra $A_1(\mathbb{C})$ the ring $A_1(\mathbb{C}; q) = \mathbb{C}\langle X, Y\rangle/\langle YX - qXY - 1\rangle$. This ring can also be described as the skew polynomial ring $\mathbb{C}[X][Y; \sigma, D_q]$, where q is a fixed complex number, σ is the automorphism of $\mathbb{C}[X]$ fixing $\mathbb{C}$ and taking X to qX, and D_q is the q-difference operator $D_q(f) = (f(qX) - f(X))/(qX - X)$, a σ-skew derivation. The prime ideal structure of the ring $A_1(\mathbb{C}; q)$ was studied in [G] for both the case in which q is not a root of unity, and the one in which q is a root of unity. Throughout this paper we will always assume that q is not a root of unity. Since the usual Weyl algebra occurs in the solution of the harmonic oscillator problem, it is natural to expect a q-analog of the Weyl algebra to arise from a q-analog of the harmonic oscillator problem.

One method of producing q-analogs has been to take representations of algebras, and to replace each integer in the representation by an appropriate "q-analog of the integer." One such "q-number" is $[n]_q = (q^n - 1)/(q - 1) = q^{n-1} + q^{n-2} + \cdots + q + 1$. Note that as an operator on the polynomial ring $\mathbb{C}[X]$, $D_q[X^n] = [n]_q X^{n-1}$. It then is easily checked that the usual formal solution of the harmonic oscillator problem follows (as in [SW, p. 48–50]), where the integer n is replaced by $[n]_q$ and d/dx is replaced by D_q; this generalization of the harmonic oscillator problem was considered by physicists M. Arik and D.D. Coon in work [AC] done well before the current interest in quantum groups. The operators a^+ and a give a faithful irreducible representation of $A_1(\mathbb{C};q)$ on the vector space $V = \{v_n \colon n = 0, 1, \ldots\}$, where $a^+ v_n = \sqrt{[n+1]_q}\, v_{n+1}$, $av_0 = 0$, $av_n = \sqrt{[n]_q}\, v_{n-1}$ (note that a^+ is the transpose of a). This gives the following proposition.

PROPOSTION 1.1: *The ring $A_1(\mathbb{C};q)$ for q not a root of unity is a primitive ring. (Note that $A_1(\mathbb{C};q)$ is not a simple ring by [G, Proposition 8.2], since $YX - XY$ is a normalizing of $A_1(\mathbb{C};q)$, and generates a proper two-sided ideal).*

The fact that $A_1(\mathbb{C};q)$ is primitive can also be obtained by noting that it is a semiprimitive G-ring, since when it is localized at the powers of $c = YX - XY$, it becomes simple [G, Theorem 8.4].

If $A_1(\mathbb{C};q)$ is filtered in the usual way, by taking X and Y to be of degree one, then the associated graded ring $\mathrm{Gr}(A_1(\mathbb{C};q))$ is $\mathbb{C}_q[X,Y]$, the "quantum plane" (the skew polynomial ring with $YX = qXY$). When a similar filtration is taken on $A_1(\mathbb{C})$, $\mathrm{Gr}(A_1(\mathbb{C}))$ is $\mathbb{C}[X,Y]$, the affine plane.

The "q-number" $[n]_q = (q^n - q^{-n})/(q - q^{-1}) = q^{n-1} + q^{n-3} + \cdots + q^{-(n-3)} + q^{-(n-1)}$ is symmetric in q and q^{-1}, and seems to be more natural to both physicists (see e.g. [B], [M]) and representation theorists (see e.g. [L1]). When one replaces n by this $[n]_q$ in the matrix representations of the operators a^+ and a (so that $a^+(v_n) = \sqrt{[n+1]_q}\, v_{n+1}$, $av_0 = 0$, $av_n = \sqrt{[n]_q}\, v_{n-1}$) one obtains matrices a,a^+, and N satisfying

$$
\begin{aligned}
&aa^+ - qa^+a = q^{-N},\\
(1)\qquad &Na^+ - a^+N = a^+,\\
&Na - aN = -a,
\end{aligned}
$$

where N is the "number operator" (the diagonal matrix with $n_{ii} = i$ for $i = 0, 1, \ldots$)(see e.g. [M, §3]). In [B] and [M] it is shown that these operators have properties analogous to those of the classical harmonic oscillator, so that the matrices a, a^+, N can be regarded as a "q-analog of the quantum harmonic oscillator." We wish to rewrite these relations by letting $L = q^{-N}$; then $Na^+ - a^+N = a^+$, so that $Na^+ = a^+(N+1)$, and therefore $g(N)a^+ = a^+g(N+1)$ holds for any polynomial $g(X)$; hence we have $q^N a^+ = a^+ q^{N+1}$, or $q^{-N}a^+ = q^{-1}a^+q^{-N}$, so that $La^+ = q^{-1}a^+L$. Similarly we have $La = qaL$. Note that under the given matrix representation for a and a^+, L is represented by the diagonal matrix with $l_{ii} = q^{-i}$ for $i = 0, 1, \ldots$ We will denote by H_q the ring generated by a,a^+,L subject to the relations:

$$(2)\qquad \begin{aligned} aa^+ - qa^+a &= L, \\ La^+ &= q^{-1}a^+L, \\ La &= qaL. \end{aligned}$$

Notice that when $q = 1$, H_q becomes $U(h)$, and hence H_q is a "q-analog" of $U(h)$. These relations can be written in "quommutator" form by defining $[x, y]_q = xy - qyx$. The ring H_q then is generated by the quommutator relations:

$$[a, a^+]_q = L, \qquad [a^+, L]_q = 0, \qquad [L, a]_q = 0.$$

The ring H_q is a skew polynomial extension of the quantum plane. Let R be the subring of H_q generated by a^+ and L, $R = \mathbb{C}[L][a^+, \sigma']$ where $\sigma'(L) = qL$ (since $a^+L = (qL)a^+$); then R is isomorphic to the quantum plane. The ring H_q is $R[a; \sigma, \delta]$, where $\sigma(a^+) = qa^+$, $\sigma(L) = q^{-1}L$, $\delta(a^+) = L$, and $\delta(L) = 0$ (since $aa^+ - (qa^+)a = L$ and $aL - (q^{-1}L)a = 0$). Goodearl [G, p.32] has noted that $\delta\sigma = q^2\sigma\delta$, and that σ,δ were used in [MS, Theorem 4.3].

It is not hard to check that the matrix representation for a, a^+, L generates an irreducible representation for H_q; this representation is not faithful since the matrices satisfy $aa^+ - q^{-1}a^+a = L^{-1}$ (note also that when $q = 1$ the matrices give a representation of the Weyl algebra, not $U(h)$). We shall see that this representation is a faithful irreducible representation of the ring satisfying the relations (2) and the additional relation $aa^+L - q^{-1}a^+aL = 1$.

There is another interesting way in which H_q arises, as pointed out to us by Susan Montgomery. There is an embedding of the usual Heisenberg algebra h into $sl(3)$ via $h = \langle e_1, e_2 \rangle$; hence it is reasonable to consider the subring of $U_q(sl(3))$ generated by E_1 and E_2 (this subring is $U_q(sl(3))^+$ in the notation of [L2]). In $U_q(sl(3))$ (using the notation of [L2], or in the notation of [S] but replacing his q by $\sqrt{q}$) E_1 and E_2 satisfy the relations:

$$(3)\qquad \begin{aligned} E_1^2E_2 + E_2E_1^2 &= (q + q^{-1})E_1E_2E_1, \\ E_2^2E_1 + E_1E_2^2 &= (q + q^{-1})E_2E_1E_2. \end{aligned}$$

Identifying a with E_1, a^+ with E_2, and L with $E_1E_2 - qE_2E_1$, we see that this subring of $U_q(sl(3))$ is isomorphic to the ring H_q described above.

Both [B] and [M] note a second interesting relationship between H_q and a quantum group. The usual Jordan–Schwinger representation of $U(\mathrm{su}(2))$ uses "two commuting harmonic oscillators" (which algebraically is equivalent to $A_2(\mathbb{R})$, the second Weyl algebra) to produce a representation of $U(\mathrm{su}(2))$ (see e.g. [SW, p. 51–52]). In [B] and [M] it is shown that two commuting q-analog harmonic oscillators can be used to produce a representation of the quantum groups $U_q(\mathrm{su}(2))$ in an analogous manner ($U_q(\mathrm{su}(2))$ has the same relations as $U_q(sl(2))$, but is defined over $\mathbb{R}$ and has a $*$-operation).

The ring $U(h)$ is a standard example of a graded Noetherian domain of global dimension three which is regular in the sense of Artin–Schelter [AS]. Recall that a graded k-algebra A is called **regular of dimension d** if (i) $\mathrm{gldim} A = d$; (ii) $GK\mathrm{dim} A < \infty$; and (iii) A is Gorenstein (i.e. $\mathrm{Ext}^q_A(k, A) = \delta_{d,q} k$). Note also that the ring $U(h)$ has a nontrivial center. All of these properties are shared by H_q.

PROPOSITION 1.2:

(1) *H_q is a graded Noetherian domain of global dimension three which is regular in the sense of [AS]. (In the terminilogy of [AS], it is of type S_1; see [AS, (8.5), p. 203]).*

(2) *If $u = aa^+ - q^{-1}a^+a$, then $Lu = uL$ is in the center of H_q; hence H_q is not a primitive ring.*

Proof: (1) Since H_q is generated by E_1 and E_2 subject to the homogeneous relations (3), it is clear that H_q is a graded ring. One can also filter H_q by taking a, a^+, L to be of degree one, and the associated graded ring $\mathrm{Gr}(H_q)$ is isomorphic to a skew polynomial ring in three indeterminates, so that H_q is a Noetherian domain of $\mathrm{gldim}(H_q) \leq 3$. As noted, H_q is (8.5) of [AS], taking $\alpha = 1$ and $a = -q^2 - q^{-2}$.

(2) One checks that $a^+u = q^{-1}ua^+$ and $ua = q^{-1}au$, so that $Lu = uL$ is central. Since the center of H_q is not a field, it is clear that H_q cannot be a primitive ring (see e.g. [O, Proposition 1]). ∎

T. Hayashi [H] has considered the ring A_q generated by a, a^+, and L with relations (2), along with the additional relation

$$aa^+L - q^{-1}a^+aL = 1, \tag{4}$$

added to obtain symmetry with q and q^{-1}. Since A_q is a factor ring of H_q, it is a Noetherian ring; we will show that A_q is a simple domain which is analogous to the Weyl algebra $A_1(\mathbb{C})$. Hayashi considered analogs $A_q(n)$ of the Weyl algebras $A_n(\mathbb{C})$ by defining $A_q(n)$ to be the ring generated by n commuting q-analog oscillators (with the additional relation (4)): namely, $A_q(n) = \mathbb{C}\langle a_i, a_i^+, L_i : 1 \leq i \leq n\rangle$ with relations:

$$
\begin{aligned}
&a_i a_j^+ = a_j^+ a_i \text{ for } i \neq j,\\
&a_i L_j = L_j a_i \text{ for } i \neq j,\\
&a_i^+ L_j = L_j a_i^+ \text{ for } i \neq j,\\
&a_i a_i^+ - q a_i^+ a_i = L_i,\\
&a_i a_i^+ - q^{-1} a_i^+ a_i = L_i^{-1},\\
&L_i a_i^+ = q^{-1} a_i^+ L_i,\\
&L_i a_i = q a_i L_i.
\end{aligned} \tag{5}
$$

Hayashi [H] showed that $A_q(n)$ could be used to produce unitary oscillator representations of $U_q(g)$ where g is a classical Lie algebra of types A and C (he defined a related q-analog of the Clifford algebra to obtain spinor representations of classical Lie algebras of types B and D). Thus from the point of view of representation theory, the ring $A_q = A_q(1)$ is analogous to the Weyl algebra.

The ring A_q described above is also the ring which plays the role of the Weyl algebra in Hodge's "quantum analog" of the Bernstein–Beilinson Theorem. Indeed, using the notation of [Ho] (but replacing q^2 by q), Hodge's ring is generated by three vector space endomorphisms T, σ, and δ of $\mathbb{C}[T]$. These endomorphisms satisfy the following relations: $\delta T - qT\delta = \sigma^{-1}$, $\delta T - q^{-1}T\delta = \sigma$, $\sigma T = qT\sigma$, and $q\sigma\delta = \delta\sigma$. It is perhaps worth noting that since $\delta(T^i) = ((q^i - q^{-i})/(q - q^{-1}))T^{i-1} = [i]_q T^{i-1}$, δ is merely the q-difference operator $\delta(f) = (f(qT) - f(q^{-1}T))/(qT - q^{-1}T)$. Hodges argues that this ring A_q arises naturally out of geometric constructions.

We shall also see that A_q shares many ring theoretic properties with the Weyl algebra. We begin by recalling that $z \in U(h)$ is central, and $A_1(\mathbb{C}) = U(h)/\langle z - 1\rangle$; we have also seen that $uL \in H_q$ is central and $A_q = H_q/\langle uL - 1\rangle$. Note, however, that when $q = 1$, we have $L^2 - 1 = 0 = (L-1)(L+1)$ in A_q so that A_q is not a domain when $q = 1$ (so, in particular, A_q does not become $A_1(\mathbb{C})$ when $q = 1$).

To prove that A_q is a simple domain we begin by describing A_q in terms of other generators. Note that (4) implies that $a(a^+L) - q^{-2}(a^+L)a = 1$, so that

A_q has a subring $\hat{A} = \mathbb{C}\langle a, a^+L\rangle \simeq A_1(\mathbb{C}; q^{-2})$, Goodearl's q-analog to the Weyl algebra discussed earlier. Let $y = a$, $x = a^+L$, and $z = L^{-1}$. Then $\hat{A} = \mathbb{C}\langle x, y\rangle$ and $A_q = \hat{A}\langle z\rangle$. Furthermore, $L^{-2}(a(a^+L) - (a^+L)a) = L^{-2}(aqLa^+ - qLa^+a) = L^{-2}(Laa^+ - qLa^+a) = L^{-1}(aa^+ - qa^+a) = L^{-1}L = 1$, so $z^2 = (yz - xy)^{-1}$. We collect some facts about this description A_q in the following lemma.

LEMMA 1.3: *Let $p = q^{-2}$. Then $A_q = \mathbb{C}\langle x, y, z\rangle$ where $yx - pxy = 1$, $z^2 = (yx - xy)^{-1} = \theta^{-1}$, $zy = q^{-1}yz$, $zx = qxz$, $\theta = yx - xy$ is a normalizing element of $\hat{A} = \mathbb{C}\langle x, y\rangle = A_1(\mathbb{C}; p)$, and $\hat{A}/\hat{A}\theta$ is a domain.*

Proof: The given relations follow from our identification of $x = a^+L$, $y = a$, and $z = L^{-1}$. The facts about $\hat{A}$ follow from [G, Proposition 8.2]. ∎

Let T be the localization of $\hat{A}$ at the powers of θ; since θ is contained in all nonzero prime ideals of $\hat{A}$ [G, Theorem 8.4], T is a simple Noetherian domain. Let $D = Q(\hat{A})$ be the total quotient ring of $\hat{A}$, and let $S = D\langle z\rangle$. Note that since z is a normalizing element of A_q, then z is a normalizing element of S, and for any $d \in D$ we have $dz = z\bar{d}$ for some $\bar{d} \in D$. Thus we have $\hat{A} \subseteq T \subseteq D \subseteq S$, and to show that A_q is a domain, it suffices to show that S is a domain. Every element of S can be written as $d_0 + d_1 z$ (since $z^2 = \theta^{-1} \in T$), and if $(d_0 + d_1 z)(e_0 + e_1 z) = 0$ then $(d_1^{-1}d_0 + z)(e_0 + z\bar{e}_1) = 0$, and hence $(d_1^{-1}d_0 + z)(e_0\bar{e}_1^{-1} + z) = 0$. This shows that if S has zero divisors, we have elements $d, e, \in D$ with $(d + z)(e + z) = 0$. Thus $de + ze + dz + z^2 = 0$, and since $z^2 \in T \subseteq D$, we have $z(e + \bar{d}) \in D$. It follows that if $e + \bar{d} \neq 0$, then $z \in D$, so $A_q \subseteq D$, and A_q is a domain. Thus suppose that $e + \bar{d} = 0$ so that $ze + dz = 0$, and $\theta^{-1} = z^2 = -de = d\bar{d}$; similarly $\theta^{-1} = \tilde{e}e$ where $ze = \tilde{e}z$. We now show that this cannot happen, and hence A_q is a domain.

THEOREM 1.4: *The ring A_q is a domain.*

Proof: By taking the appropriate inverses, it follows from the remarks above that we can assume that $\theta = -gf = \bar{f}f = g\tilde{g}$, where $\theta = yx - xy \in \hat{A} = A_1(\mathbb{C}; p)$, $fz = z\bar{f}$, $zg = \tilde{g}z$, $\bar{f} = -g$, and $f = -\tilde{g}$ for $f, \bar{f}, g, \tilde{g} \in D = Q(\hat{A})$. We write $f = a_1 s_1^{-1} = t_1^{-1}b_1$ and $g = a_2 s_2^{-1} = t_2^{-1}b_2$ for $a_i, b_i, s_i, t_i \in \hat{A}$. Since θ is a normalizing element of the Noetherian ring $\hat{A}$, we can assume that if $a_i \in \hat{A}\theta$ then $s_i \notin \hat{A}\theta$ (and similarly with b_i and t_i). The relation $\theta = -gf$ gives $t_2\theta s_1 = -b_2 a_1$; since $\hat{A}/\hat{A}\theta$ is a domain, either $a_1 \in \hat{A}\theta$ or $b_2 \in \hat{A}\theta$.

Without loss of generality we can assume that $a_1 \in \hat{A}\theta$, and hence $b_1 s_1 = t_1 a_1 \in \hat{A}\theta$; since $s_1 \notin \hat{A}\theta$ we have $b_1 \in \hat{A}\theta$, and hence $t_1\tilde{g} = -t_1 f = -b_1 \in \hat{A}\theta$.

Let $t_1\tilde{g} = \theta u$ for $u \in \hat{A}$; then $\tilde{g}z = zg$ implies that $t_1 zg = t_1\tilde{g}x = \theta uz = \theta z\bar{u} = z\theta\bar{u}$ for some $\bar{u} \in \hat{A}$. Since $t_1 z = z\bar{t}_1$ for $\bar{t}_1 \in \hat{A}$, and since z is invertible in $D\langle z\rangle$, we have $\bar{t}_1 g = \theta\bar{u} \in \hat{A}\theta$; hence $\bar{t}_1 a_2 = \bar{t}_1 g s_2 = \theta\bar{u}s_2 \in \hat{A}\theta$. We claim that $\bar{t}_1 \notin \hat{A}\theta$, for if $\bar{t}_1 = \theta w$ for $w \in \hat{A}$, then $t_1 z = z\bar{t}_1 = z\theta w = \theta z w = \theta\tilde{w}z$ for $\tilde{w} \in \hat{A}$, and hence $t_1 = \theta\tilde{w} \in \hat{A}\theta$, a contradiction. Hence $a_2 \in \tilde{A}\theta$ and $b_2 s_2 = t_2 a_2 \in \tilde{A}\theta$; since $s_2 \notin \tilde{A}\theta$ we have $b_2 \in \tilde{A}\theta$. Therefore we have $b_2 a_1 \in \tilde{A}\theta^2$, so that $b_2 a_1 = -t_2\theta s_1 = -t_2 s_1'\theta = r\theta^2$ for some $s_1', r \in \hat{A}$; thus $t_2 s_1' \in \tilde{A}\theta$, which implies that $s_1' \in \tilde{A}\theta$. Since $\theta s_1 = s_1'\theta \in \tilde{A}\theta^2$ we have $s_1 \in \tilde{A}\theta$, a contradiction. ∎

It is now not difficult to see that A_q is a simple domain.

PROPOSITION 1.5: *The ring A_q is a simple Noetherian domain with center $\mathbb{C}$, and A_q is not isomorphic to $A_1(\mathbb{C})$.*

Proof: Let $0 \neq I$ be an ideal of A_q and take any $0 \neq c \in I$. Then $T\langle c\rangle$ is a Noetherian T-module so for some n, $c^n = t_0 + t_1 c + \cdots + t_{n-1}c^{n-1}$; if n is chosen to be the minimal such n, then $t_0 \neq 0$ and $t_0 \in I \cap T$. Since T is simple we have $1 \in I \cap T$ and hence $I = A_q$. The center of A_q must be a field, and since A_q is an affine $\mathbb{C}$-algebra, and hence a countable dimensional vector space over $\mathbb{C}$, the center must be $\mathbb{C}$.

The final remark holds since the only invertible elements of $A_1(\mathbb{C})$ are in $\mathbb{C}$, while A_q has $L \notin \mathbb{C}$ which is an invertible element of A_q. ∎

As we noted earlier, although the representation $a^+(v_n) = \sqrt{[n+1]_q}v_{n+1}$, $a(v_n) = \sqrt{[n]_q}v_{n-1}$ $av_0 = 0$, $L(v_n) = q^{-n}v_n$ is an irreducible representation of H_q on $V = \{v_i \colon i \in \mathbb{N}\}$, it is not a faithful representation since $(aa^+ - q^{-1}a^+a)L - 1$ acts as 0 on V. Since $A_q = H_q/\langle(aa^+ - q^{-1}a^+a)L - 1\rangle$ is a simple ring, it follows that $\operatorname{annih}_{H_q}V = H_q((aa^+ - q^{-1}a^+a)L - 1)$.

We next note that A_q has the same Krull dimension and Gelfand-Kirillov dimension as the usual Weyl algebra $A_1(\mathbb{C})$.

PROPOSITION 1.6: *The ring A_q has Gelfand-Kirillov dimension 2 and right (left) Krull dimension 1. Furthermore all right ideals of A_q can be generated by at most two elements, and no subfield of D_q, the quotient division ring of A_q, can have transcendence degree greater than 1 over $\mathbb{C}$.*

Proof: The regular ring H_q has $GK\dim H_q = 3$, so $GK\dim A_q \leq 2$ since A_q is a factor ring of H_q by a central element. If the $GK\dim$ of A_q were 1, A_q would be a P.I. ring, in fact module-finite over its center [SmW]. Hence $GK\dim A_q = 2$.

Since A_q is a finite normalizing extension of T, then the Krull dimension of A_q is the same as the Krull dimension of T (see e.g. [MR, 10.1.11(ii)]), and by an argument such as ([MR, 6.6.15]), using the fact that T is simple it can be shown that the Krull dimension of T is 1. The bound on the number of generators of right ideals follows from [St], and the final remark follows as in ([MR, 6.6.18]). ∎

The constructions we have discussed can be iterated. Inductively one can iterate Goodearl's construction and define $A_n(\mathbb{C}, q) = A_1(A_{n-1}(\mathbb{C}; q); q)$ which has generators $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and relations $y_i x_i - q x_i y_i = 1$ for all i, $x_i x_j = x_j x_i$, $y_i y_j = y_j y_i$ for all i, j, and $x_i y_j = y_j x_i$ for all $i \neq j$. It is not hard to show that $u_i = y_i x_i - x_i y_i$ are normalizing elements of $A_n(\mathbb{C}, q)$, that $A_n(\mathbb{C}, q)/\langle u_j \rangle$ are domains, and that $A_n(\mathbb{C}, q)_S$ is a simple ring, where S is the denominator set $S = \{u_1^{i_1} \cdots u_n^{i_n} : i_j \in \mathbb{N}\}$. Then $A_q(n) = \hat{A}\langle z_1, \ldots, z_n \rangle$ where $\hat{A} = A_n(\mathbb{C}; p)_S$ and the z_i are normalizing elements of $A_q(n)$ with $z_i^2 = u_i^{-1}$. Inductive arguments as above show that $A_q(n)$ is a simple Noetherian domain with Krull dimension n and Gelfand–Kirillov dimension $2n$. It is also not hard to check that $A_q(n) = A_q \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} A_q$, which gives another way of proving that $A_q(n)$ is a simple ring.

Next we shall compute the prime spectrum of H_q. We have seen that if $u = aa^+ - q^{-1}a^+a$, then $\langle uL - 1 \rangle$ is a maximal ideal of H_q. Recall that if h is any finite dimensional complex nilpotent Lie algebra, then the only primitive homomorphic images of $U(h)$ are the Weyl algebras $A_n(\mathbb{C})$ [D, Theorem 4.7.9]; hence primitive ideals are maximal ideals. Note that L is a normalizing element of H_q, and that $H_q/\langle L \rangle$ is isomorphic to the quantum plane, a primitive (but not simple) ring, so that H_q has primitive ideals which are not maximal (this answers a question of [S, p. 40]); we shall see that $\langle L \rangle$ is the only non-maximal primitive ideal of H_q.

Gabber [Ga] has shown that when g is a complex solvable Lie algebra, then $U(g)$ has the catenary property. We will see from the propostion below that

$$0 \subsetneqq \langle L \rangle \subsetneqq \langle L, a \rangle \subsetneqq H_q \quad \text{and} \quad 0 \subsetneqq \langle uL - 1 \rangle \subsetneqq H_q$$

are saturated chains of prime ideals, so that H_q does not satisfy the catenary property.

PROPOSITION 1.7: *If P is a nonzero prime ideal of H_q, then:*

(1.) *if $L \in P$, then $P/\langle L\rangle$ is a prime ideal of the quantum plane $\mathbb{C}[a^+][a;\sigma]$, whose prime ideal structure was described in [I]. The prime ideal $\langle L\rangle$ is primitive; if $P \supsetneq \langle L\rangle$ then P must contain either a^+ or a, and hence P is not primitive unless it is maximal.*

(2.) *if $L \notin P$, then $P = \langle uL - \alpha\rangle$ for some $\alpha \in \mathbb{C}$. In this case P is a maximal ideal of H_q.*

Proof: We view H_q as $R[a;\sigma,\delta]$ where $R = \mathbb{C}[L][a^+,\sigma']$, the quantum plane. If $P \cap R$ is nonzero, then since any ideal of R contains a power of a^+L, we have $(a^+)^iL^i \in P$. Since $a(a^+)^iL^i \in P$ and $(a^+)^iL^ia \in P$ we have $a(a^+)^iL^i - (a^+)^iq^iaL^i \in P$. Using the relation $a(a^+)^n - q^n(a^+)^na = [n]_q(a^+)^{n-1}L$, where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, we have

$$
\begin{aligned}
&a(a^+)^{i-1}a^+L^i - q^i(a^+)^{i-1}(a^+a)L^i = \\
&\qquad (q^{i-1}(a^+)^{i-1}a + [i-1]_q(a^+)^{i-2}L)a^+L^i - q^i(a^+)^{i-1}(a^+a)L^i = \\
&\qquad q^{i-1}(a^+)^{i-1}(aa^+ - qa^+a)L^i + [i-1]_q(a^+)^{i-2}La^+L^i = \\
&\qquad q^{i-1}(a^+)^{i-1}L^{i+1} + [i-1]_q(q^{-1})(a^+)^{i-1}L^{i+1} = \\
&\qquad (q^{i-1} + ((q^{i-1} - q^{1-i})q^{-1}/(q - q^{-1})))(a^+)^{i-1}L^{i+1} = \\
&\qquad [i]_q(a^+)^{i-1}L^{i+1} \in P.
\end{aligned}
$$

Hence inductively one can show that $L^{2i} \in P$, and since L is a normalizing element and P is a prime ideal, we have that $L \in P$, completing (1).

Next suppose that $P \cap R = 0$. We first note that $S = \{(a^+)^iL^j\colon i,j \in \mathbb{N}\}$ is an Ore set, since L is a normalizing element of H_q and $a(a^+)^2 = a^+(qaa^+ + q^{-1}L)$. Localizing at this Ore set we get the ring $T = Q[a;\sigma,\delta]$, where $Q = \mathbb{C}_q[L, a^+, L^{-1}, (a^+)^{-1}]$ is a simple ring [MP], and σ and δ have been extended to Q [G, Lemma 1.3].

Note that δ is an inner σ-derivation on Q, $\delta = \delta_r$ for $r = (1-q^2)^{-1}L(a^+)^{-1}$, since $\delta(L) = (1-q^2)^{-1}L(a^+)^{-1}L - q^{-1}L(1-q^2)^{-1}L(a^+)^{-1} = (1-q^2)^{-1}q^{-1}L^2(a^+)^{-1} - (1-q^2)^{-1}q^{-1}L^2(a^+)^{-1} = 0$, and $\delta(a^+) = (1-q^2)^{-1}L(a^+)^{-1}(a^+) - qa^+(1-q^2)^{-1}L(a^+)^{-1} = (1-q^2)^{-1}L - q(1-q^2)^{-1}qL = (1-q^2)^{-1}(1-q^2)L = L$. Hence by [G, Lemma 1.5], $T = Q[\theta;\sigma]$ where $\theta = a - L(a^+)^{-1}(1-q^2)^{-1}$. Notice that $q^2(q^2-1)^{-1}u(a^+)^{-1} = \theta$, so that $u = c\theta a^+$ for some $c \in \mathbb{C}$, and that $L\theta = q\theta L$ and $a^+\theta = q^{-1}\theta a^+$.

Let $P' = PT$ be the extended prime ideal of T, $P' \subsetneq T$. Let $g(\theta) = \theta^n + q_{n-1}\theta^{n-1} + \cdots + q_1\theta + q_0$ be a nonzero element of P' with minimal degree (since

Q is simple, $g(\theta)$ can be chosen to be monic). We have that $Lg(\theta)-q^n g(\theta)L \in P'$, and hence we have an element in P' of smaller degree unless $Lg(\theta)-q^n g(\theta)L = 0$, in which case $Lq_k\theta^k = q^n q_k\theta^k L$ for all k, or $Lq_k = q^{n-k}q_k L$ for all k. Similarly $q^{n-k}a^+q_k = q_k a^+$ for all k. We claim that these two conditions force $q_k = \alpha_k(L^{-1})^{n-k}((a^+)^{-1})^{n-k} = \beta_k(L^{-1}(a^+)^{-1})^{n-k}$ for some $\alpha_k, \beta_k \in \mathbb{C}$. Indeed, let $q_k = \sum_i\sum_j \alpha_{ijk}L^i(a^+)^j$ for $i,j \in \mathbb{Z}$. Now $Lq_k = \sum_i\sum_j a_{ijk}L^{i+1}(a^+)^j$ and $q^{n-k}q_kL = q^{n-k}\sum_i\sum_j \alpha_{ijk}q^jL^{i+1}(a^+)^j$; hence the only value of j with $a_{ijk} \neq 0$ is $j = k-n$. Thus $q_k = \sum_i \alpha_{ik}L^i((a^+)^{-1})^{n-k}$, $q_ka^+ = \sum_i \alpha_{ik}L^i((a^+)^{-1})^{n-k-1}$, and $q^{n-k}a^+q_k = \sum_i \alpha_{ik}q^{n-k}q^iL^i((a^+)^{-1})^{n-k-1}$, so that the only value of i with $a_{ik} \neq 0$ is $i = k-n$, establishing the claim. Let $t = L^{-1}(a^+)^{-1} \in Q$, and notice that $t\theta = L^{-1}(a^+)^{-1}\theta = L^{-1}q\theta(a^+)^{-1} = \theta L^{-1}(a^+)^{-1} = \theta t$. Hence $g(\theta) = \theta^n + \beta_{n-1}t\theta^{n-1} + \beta_{n-2}t^2\theta^{n-2} + \cdots + \beta_1t^{n-1}\theta + \beta_0t^n = (\theta+\gamma_1t)\cdots(\theta+\gamma_nt)$ for some $\gamma_i \in \mathbb{C}$ since $\theta t = t\theta$. Furthermore $\theta + \gamma_it$ is a normalizing element of T $((\theta+\gamma_it)L = q^{-1}L(\theta+\gamma_it)$ and $(\theta+\gamma_it)a^+ = qa^+(\theta+\gamma_it))$, and $g(\theta) \in P'$ a prime ideal of T, so therefore $\theta + \gamma_iL^{-1}(a^+)^{-1} \in P'$ or $\theta a^+L + \gamma_i \in P'$, and $uL + \gamma_i^* \in P' \cap H_q = P$ for some $\gamma_i^* \in \mathbb{C}$.

One checks that for $\gamma^* \neq 0$, $H_q/\langle uL+\gamma^*\rangle \simeq H_q/\langle uL-1\rangle$ under the map that takes a to $a/\sqrt[4]{-\gamma^*}$, a^+ to $a^+/\sqrt[4]{-\gamma^*}$, and L to $L/\sqrt{-\gamma^*}$, so $\langle uL+\gamma^*\rangle$ is a maximal ideal of H_q. ∎

Curtright and Zachos [CZ] have proposed as a "quantum Virasoro algebra" the free ring $\mathbb{C}\langle x_i; i \in \mathbb{Z}\rangle$ modulo the relations:

(6) $$q^{l-k}X_kX_l - q^{k-l}X_lX_k = [l-k]_qX_{l+k} \text{ for } l,k \in \mathbb{Z}.$$

It is an open question whether this ring V_q has a Hopf structure. However Chaichian, Kulish, and Lukierski [CKL] have shown that within $Q(A_q)$ the elements $Y_n = L(a^+)^{n+1}a = q^{-(n+1)}(a^+)^{n+1}(La)$ for $n \in \mathbb{Z}$, satisfy the relations (6), and hence the q-analog oscillator of Hayashi gives a representation of the "quantum Virasoro algebra" of Curtright and Zachos in an analogous way that a localization of the Weyl algebra gives a representation of the usual Virasoro algebra.

Dean and Small [DS] have shown how to obtain irreducible representations of the Virasoro algebra from irreducible representations of a localization of the Weyl algebra [DS, Theorem 6]; an analogous result holds here. Let V be the subring of $Q(A_q)$ generated by $\{Y_n\}$ and 1 over $\mathbb{C}_q$; then V is the subring generated by 1 and $\{(a^+)^n(La)\}$. In A_q we have the relation $Laa^+ - q^{-1}La^+a = 1$, so $(La)a^+ - q^{-2}a^+(La) = 1$, and La and a^+ generate a copy of $A_1(\mathbb{C}; q^{-2})$. Notice that the powers of a^+ form an Ore set in this subring since $(La)(a^+)^2 = (a^+)(1 + q^{-2}(La)a^+)$. Let L_1 be the localization of $A_1(\mathbb{C}; q^{-2})$ at the powers of a^+, and notice that L_1 is a Noetherian ring. The subring V of L_1 is $V = \mathbb{C} + L_1(La) = \prod_{L_1}(L_1(La))$, the idealizer in L_1 at the left ideal $L_1(La)$. Notice that $L_1(La)L_1 = L_1$, so that $L_1(La)$ is a generative left ideal of L_1; also note that $L_1(La)$ is a maximal left ideal of L_1. Hence we obtain the following analog of [DS, Theorem 6].

PROPOSITION 1.8: *Let M be an irreducible L_1-module, $M \neq L_1/L_1a$. Then M is an irreducible V_q-module. The irreducible L_1-module L_1/L_1a has the following composition series when regarded as a V_q-module:*

$$L_1 \supseteq V + L_1a \supseteq L_1a.$$

As one example, the irreducible L_1-module $L_1/L_1(a^+ + 1)$ is an irreducible V_q-module.

2. Another q-analog of $U(h)$

One might expect that all q-analogs of an enveloping algebra are quite similar. Hence we consider next the ring H'_q generated by a,a^+, and L with the slightly different relations:

$$\begin{aligned} &aa^+ - qa^+a = L \\ (7)\qquad &aL = qLa \\ &La^+ = qa^+L \end{aligned}$$

When $q = 1$, H'_q becomes $U(h)$, so that H'_q is another q-analog of $U(h)$. We shall see the structure of H'_q is quite different than that of H_q. If H'_q is filtered by taking a, a^+, and L to be of degree one, then the associated graded ring $\mathrm{Gr}(H'_q) \cong \mathbb{C}_q[X_1, X_2, X_3]$, quantum 3-space (with $X_1 = a$, $X_2 = L$, and $X_3 = a^+$). Since $\mathbb{C}_q[X_1, X_2, X_3]$ has center $\mathbb{C}$, H'_1 has center $\mathbb{C}$, and hence H'_q is not isomorphic to H_q. It is not difficult to check the following:

PROPOSITION 2.1: *The ring H'_q is the skew polynomial ring, $H'_q = R[a;\sigma,\delta]$ where $R = \mathbb{C}[a^+][L;\sigma]$ with $\sigma(a^+) = qa^+$ (i.e. $R \simeq \mathbb{C}_q[X_1, X_2]$, the quantum plane); the automorphism σ is extended to R by defining $\sigma(L) = qL$, and the σ-skew derivation δ is defined by $\delta(a^+) = L$ and $\delta(L) = 0$.*

S. Amitsur [A] showed that if A is a simple k-algebra (more generally if A has no δ-stable ideals), with the characteristic of k equal to zero, and if δ is not an inner derivation, then $A[X;\delta]$ is a simple ring. In [G], K. Goodearl introduced the concept of an "s-skew derivation" (a triple (σ,δ,s), where δ is a σ-derivation, and $\sigma\delta = s\delta\sigma$ for some central element s of A with $\sigma(s) = s$ and $\delta(s) = 0$). Amitsur's result extends to quantized skew derivations as follows:

PROPOSITION 2.2: *Let A be a $\mathbb{Q}$-algebra and $S = A[X;\sigma,\delta]$ where σ is an automorphism of A. If A is a simple ring, δ is an s-skew derivation $\sigma\delta = s\delta\sigma$ for $s = 1$ or s a non-root of unity, and δ is not an inner σ-derivation, then S is a simple ring.*

Proof: We use argument similar to that used in proving Amitsur's theorem (see e.g. [MR, 1.8.4, p. 34–35]). Let $0 \neq I$ be a proper ideal of S. It is not hard to check that if I_n is defined to be the leading coefficients of elements of I of degree $\leq n$, then I_n are ideals of A. Choose the least n with $I_n \neq 0$ so that $I_n = A$. If $n = 0$ then $I = S$. If $n > 0$ then there is a monic polynomial $f(x) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in I$. For any $a \in A$ let $b \in A$ be such that $(\sigma \circ \cdots \circ \sigma)(b) = \sigma^n(b) = a$. We have:

$$\begin{aligned} af - bf = &(aa_{n-1} - a_{n-1}\sigma^{n-1}(b) - \delta\sigma^{n-1}(b) - \sigma\delta\sigma^{n-2}(b) - \sigma^2\delta\sigma^{n-3}(b) - \\ &\cdots - \sigma^{n-1}(\delta(b)))X^{n-1} + \text{lower degree terms.} \end{aligned}$$

Since $\sigma\delta = s\delta\sigma$, this becomes

$$\begin{aligned} af - fb = &(aa_{n-1} - a_{n-1}\sigma^{n-1}(b) - (1 + s + \cdots + s^{n-1})\delta(\sigma^{n-1}(b)))X^{n-1} \\ &+ \text{lower degree terms.} \end{aligned}$$

Using the notation $[n]_s = 1+s+\cdots+s^{n-1}$, by the minimality of n we have that $aa_{n-1} - a_{n-1}\sigma^{n-1}(b) - [n]_s\delta(\sigma^{n-1}(b)) = 0$ for any $a \in A$, and hence $a(a_{n-1}/[n]_s) - (a_{n-1}/[n]_s)\sigma^{n-1}(b) = \delta(\sigma^{n-1}(b))$, or $\sigma(x)(a_{n-1}/[n]_s) - (a_{n-1}/[n]_s)x = \delta(x)$ for any $x \in A$. Hence $\delta = \delta_{(-a_{n-1}/[n]_s)}$, a σ-inner derivation. ∎

We now use this proposition to show that H'_q is a primitive ring.

PROPOSITION 2.3: *H'_q is a primitive ring.*

Proof: We have seen $H'_q = R[a;\sigma,\delta]$ where $R = \mathbb{C}_q[a^+, L]$ is the quantum plane. Notice that L is a normalizing element of H'_q, and since $a(a^+)^2 = a^+(qaa^+ + qL)$, we can localize H'_q at the Ore set $\{L^i(a^+)^j\}$ obtaining the ring $S = \mathbb{C}_q[a^+, L, (a^+)^{-1}, L^{-1}][a;\sigma,\delta]$, where σ and δ have been extended to the simple ring $A = \mathbb{C}_q[a^+, L, (a^+)^{-1}, L^{-1}]$. We claim that δ is not an inner derivation on A. Indeed, suppose that $\delta = \delta_g$ for $g = g(a^+, L) = \sum_{i=-n}^{n}\sum_{j=-n}^{n} \alpha_{ij}(a^+)^i L^j$; then $L = \delta(a^+) = ga^+ - qa^+g$ and $0 = \delta(L) = gL - qLg$. But $gL = \sum\sum \alpha_{ij}(a^+)^i L^{j+1}$ and $qLg = \sum\sum \alpha_{ij}q^{i+1}(a^+)^i L^{j+1}$ implies that the only value of i with $\alpha_{ij} \neq 0$ is $i = -1$. Then $g = \sum_{j=-n}^{n} \alpha_j (a^+)^{-1} L^j$, and so

$$\begin{aligned} L = & ga^+ - qa^+g = \sum \alpha_j (a^+)^{-1} L^j a^+ - \sum q\alpha_j L^j = \\ & \sum \alpha_j q^j (a^+)^{-1} a^+ L^j - \sum q\alpha_j L^j = \\ & \sum \alpha_j (q^j - q) L^j, \end{aligned}$$

which has no solution for α_j. Thus δ is not an inner σ-derivation, and S is a simple ring by the previous proposition.

If P is a nonzero prime ideal of H'_q, then $(a^+)^i L^j \in P$ for some $i, j \in \mathbb{N}$. Hence $a(a^+)^i L^j - q^{i+j}(a^+)^i L^j a \in P$, or $a(a^+)^i L^j - q^i (a^+)^i a L^j \in P$. Using the identity $a(a^+)^i - q^i(a^+)^i a = iq^{i-1}(a^+)^{i-1}L$, we obtain $(a^+)^{i-1}L^{j+1} \in P$, and hence, inductively, $L^{i+j} \in P$. Since L is a normalizing element of H'_q, then any nonzero prime ideal of H'_q contains L, so that H'_q is a G-ring. Since H'_q is a graded domain, it is semiprimitive, and hence it must be a primitive ring. ∎

We conclude by noting further properties of H'_q, properties shared by $U(h)$.

PROPOSITION 2.4: *H'_q is a graded Noetherian domain of global dimension three which is regular in the sense of [AS]. (In the terminology of [AS], it of type S_1; see [AS, (8.5), p. 203].)*

Proof: The first conditions follow as in Proposition 1.2. To see the regularity, first note that H'_q is generated by a and a^+ to the relations:

$$a^2a^+ - qaa^+a = qaa^+a - q^2a^+a^2$$

and

$$a(a^+)^2 - qa^+aa^+ = qa^+aa^+ - q^2(a^+)^2a$$

or equivalently,

$$a^2a^+ + q^2(a^+)^2a^2 = 2qaa^+a$$

and

$$a(a^+)^2 + q^2(a^+)^2a = 2qa^+aa^+,$$

which is (8.5) of [AS] (with $\alpha = q^2$ and $a = -2q$). ∎

References

[A] S. A. Amitsur, *Derivations in simple rings,* Proc. London Math. Soc.(3) **7** (1957), 87–112.

[AC] M. Arik and D. D. Coon, *Hilbert spaces of analytic functions and generalized coherent states,* J. Math. Phys. **17** (1976), 524–527.

[AS] M. Artin and W. Schelter, *Graded algebras of global dimension 3,* Advances in Math. **66** (1987), 171–216.

[B] L. C. Biedenharn, *A q-boson realization of the quantum group $SU_q(2)$ and the theory of q-tensor operators,* J. Phys. **A 22, No. 18** (1989), L873–L878.

[CKL] M. Chaichian, P. Kulish, and J. Lukierski, *q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras,* Phys. Lett. B **237** (1990), 401.

[CZ] T. L. Curtright and C. K. Zachos, *Deforming maps for quantum algebras,* Phys. Lett. B **243** (1990), 237–244.

[DS] C. Dean and L. W. Small, *Ring theoretic aspects of the Virasoro algebra,* Comm. Alg. **18** (1990), 1425–1431.

[D] J. Dixmier, *Représentations irréductibles des algèbres de Lie nilpotente,* An. Acad. Brasil. Ci. **35** (1963), 491–519.

[Ga] O. Gabber, *Equidimensionalité de la variété caractéristique, Exposé de O. Gabber redigé par T. Levasseur,* preprint, University of Paris VI.

[G] K. R. Goodearl, *Prime ideals in skew polynomial rings and quantized Weyl algebras,* J. Algebra **150** (1992), 324–377.

[H] T. Hayashi, *Q-analogues of Clifford and Weyl algebras - Spinor and oscillator representations of quantum enveloping algebras,* Commun. Math. Phys. **127** (1990), 129–144.

[Ho] T. J. Hodges, *Ring-theoretical aspects of the Bernstein–Beilinson theorem,* in *Noncommutative Ring Theory* (Athens, OH, 1989), Lecture Notes in Mathematics **1448**, Springer–Verlag, Berlin, 1990, pp. 155–163.

[I] R. Irving, *Prime ideals in Ore extensions over commutative rings,* J. Algebra **56** (1979), 315–342.

[M] A. J. Macfarlane, *On q-analogues of the quantum harmonic oscillator and the quantum group $SU_q(2)$,* J. Phys. **A 22, No. 21** (1989), 4581–4588.

[MP] J. C. McConnell and J. J. Pettit, *Crossed products and multiplicative analogues of Weyl algebras,* J. London Math. Soc.(2) **38** (1988), 47–55.

[MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings,* Wiley–Interscience, New York, 1987.

[MS] S. Montgomery and S. P. Smith, *Skew derivations and $U_q(sl(2))$,* Israel J. Math **72** (1990), 158–166.

[Mo] H. Morikawa, *On ς_n-Weyl algebra $W_r(\varsigma_n, Z)$,* Nagoya Math J. **113** (1989), 153–159.

[O] A. I. Ooms, *On Lie algebras having a primitive enveloping algebra,* J. Algebra **32** (1974), 488–500.

[L1] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras,* Adv. Math **70** (1988), 237–249.

[L2] G. Lusztig, *Quantum groups at roots of 1,* Geom. Ded (1990).

[R] M. Rosso, *Finite dimensional representations of the quantum analog of the enveloping algebra of a complex semisimple Lie algebra,* Commun. Math. Phys. **117** (1988), 581–593.

[SW] D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras With Applications to Physics, Geometry, and Mechanics,* Springer–Verlag, New York, 1986.

[SmW] L. W. Small and R. B. Warfield, Jr., *Prime affine algebras of Gelfand–Kirillov dimension one,* J. Algebra **91** (1984), 386–389.

[S] S. P. Smith, *Quantum groups: an introduction and survey for ring theorists,* in *Noncommutative Rings* (S. Montgomery and L. Small, eds.), M.S.R.I. Publ. # **24** Springer–Verlag, Berlin, 1992, pp. 131–178.

[St] J. T. Stafford, *Completely faithful modules and ideals of simple Noetherian rings,* Bull. London Math Soc. **8** (1976), 168–173.